Optimization of field-dependent nonperturbative renormalization group flows

Léonie Canet
Laboratoire de Physique Théorique et Hautes Énergies, Universités Paris VI – Pierre et Marie Curie, Paris VII – Denis Diderot, 2 place Jussieu, 75251 Paris cedex 05, France
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We investigate the influence of the momentum cutoff function on the field-dependent nonperturbative renormalization group flows for the three-dimensional Ising model, up to the second order of the derivative expansion. We show that, even when dealing with the full functional dependence of the renormalization functions, the accuracy of the critical exponents can be simply optimized, through the principle of minimal sensitivity, which yields $\nu = 0.628$ and $\eta = 0.044$.

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Many phenomena in high-energy as well as in statistical physics cannot be tackled using perturbative methods, either because they come under the strong coupling regime of their field theory or because they appear genuinely nonperturbative like, for instance, confinement in QCD or phase transitions induced by topological defects. The Wilsonian renormalization group concept, resting on progressive integration of high-energy fluctuations [1], has rooted the nonperturbative renormalization group (NPRG) formalism. The latter has arousen a renewed attention, since the idea emerged [2, 3, 4] to apply this procedure to the Legendre transform Γ of the free energy — the Gibbs free energy — rather than to the Hamiltonian. This method has allowed to broach notoriously difficult problems, such as the abelian Higgs model relevant for superconductivity [5], the phase diagram of He_3 [6], the Gross-Neveu model in three dimensions [7, 8], frustrated magnets [9] or reaction-diffusion processes [10, 11] (see [12] and [13] for reviews).

The NPRG formalism consists in building a sequence of running effective average actions Γ_k, which are infrared regulated by a momentum cutoff and thus only include fluctuations with momenta larger than the running scale k. Γ_k continuously interpolates between the microscopic action $\Gamma_{k=\Lambda} = S$ and the Gibbs free energy $\Gamma_{k=0} = \Gamma$. Its flow with k is governed by an exact equation [14]:

$$\partial_k \Gamma_k(\psi) = \frac{1}{2}\mathrm{Tr}\left\{[\Gamma_k^{(2)} + R_k]^{-1}\,\partial_k R_k\right\}, \qquad (1)$$

where $\Gamma_k^{(2)}$ is the full inverse propagator and R_k is the infrared cutoff function [38].

Though Eq. (1) is exact, hence preserving all nonperturbative features of the model, it is functional and, obviously, cannot be solved exactly. Any practical calculation requires to truncate Γ_k, most commonly through a derivative expansion [14]. Furthermore, when the number of invariants or of renormalization functions grows, one often resorts to an additional truncation to simplify the numerical task, which consists in field expanding the different renormalization functions ([14] and see below).

Naturally, these expansions raise questions as for their convergence and accuracy, since these properties entirely condition the reliability of the method, and many works have thus been devoted to their study [15, 16, 17, 18, 19, 20, 21, 22, 23]. These issues have appeared intimately related to that of the influence of the cutoff function R_k. Although the exact solution $\Gamma = \lim_{k\to 0}\Gamma_k$ does not depend on R_k, any truncation breaks this invariance. This suggests that the choice of the cutoff may be optimized, and various criteria have been proposed, such as the optimization of the rapidity of the convergence of the field expansion [19, 24], the maximization of the "gap" of the propagator [20, 25, 26, 27] or the principle of minimal sensitivity (PMS) [28, 29, 30]. All these analyses rely on the field expansion of the renormalization functions.

However, a field expansion cannot always be performed. First, it is meaningless in dimensions where the field canonical dimension vanishes, since all field powers then become equally relevant. More importantly, some phenomena require, in principle, a functional description, for instance if the effective potential develops non-analycities, which can occur for some disordered systems [31, 32, 33] and continuous growth models [34]. Since the optimization is crucial to control approximations, one has to dispose of an efficient procedure to deal with the field-dependent renormalization functions. However, this had so far never been investigated as it involves a sizeable numerical task. In this article, we show that, even without truncating the field-dependence, the PMS allows to simply optimize the choice of R_k, and we provide the corresponding optimal critical exponents for the three-dimensional Ising model, up to the second order of the derivative expansion.

Let us first draw the framework of this calculation. The derivative expansion physically rests on the assumption that the long-distance physics — corresponding to the low-energy modes ($q \to 0$) — is well captured by the lowest order derivative terms. Hence, for the Ising model, the *ansatz* of Γ_k at order ∂^2 writes [14]:

$$\Gamma_k = \int d^d x \left\{U_k(\rho) + \frac{1}{2}Z_k(\rho)\,(\nabla\psi)^2\right\}, \qquad (2)$$

where $\rho = \psi^2/2$ is the $\mathbb{Z}_2$ invariant. The running poten-

tial $U_k(\rho)$ describes the physics associated with uniform field configurations while the field renormalization function $Z_k(\rho)$ renders the effect of slowly varying fields. The flow equations of the running functions U_k and Z_k follow from Eq. (1), respectively through its evaluation at a uniform field configuration and through isolating the q^2 dependence of its second functional derivative with respect to $\psi(q)$ and $\psi(-q)$ [14]. The running anomalous dimension is then defined by [12]:

$$\eta_k = -\partial_t \ln Z_{k,0}, \tag{3}$$

where $t = \ln(k/\Lambda)$ and $Z_{k,0} \equiv Z_k(\rho_0)$ is the field renormalization at the running minimum $\rho_0(k)$ of the potential. We introduce the dimensionless and renormalized variables $\tilde{\rho} = Z_{k,0}\, k^{2-d}\, \rho$, $u_k = k^{-d}\, U_k$ and $z_k = Z_{k,0}^{-1}\, Z_k$, which are more convenient for the search of fixed points.

The field expansion would then consist in expanding u_k and z_k in powers of the invariant $\tilde{\rho}$ around the running minimum $\tilde{\rho}_0(k)$ of the potential — the latter point conveying nice convergence properties [19, 20] — which would write for a generic function h_k:

$$h_k(\tilde{\rho}) = \sum_{n=0}^{p_h} \frac{1}{n!}\, h_{k,n}\, (\tilde{\rho} - \tilde{\rho}_0)^n. \tag{4}$$

We here do not proceed to such an expansion. This expression is given for completeness since we shall compare, in the following, our results with those obtained in [29] ensuing from a field truncation of the renormalization functions u_k and z_k to the tenth and ninth order respectively — that is $p_u = 10$ and $p_z = 9$ in Eq. (4).

To determine the critical exponents of the Ising model, we numerically integrate the flow equations $\partial_t u_k(\tilde{\rho})$ and $\partial_t z_k(\tilde{\rho})$ from an initial microscopic scale Λ — at which the potential is quartic $u_\Lambda(\tilde{\rho}) = \lambda/2\, (\tilde{\rho} - \tilde{\rho}_0)^2$ and the field renormalization is constant $z_\Lambda(\tilde{\rho}) = 1$ — to the physical scale $k \to 0$. For large values of the initial parameter $\tilde{\rho}_0(\Lambda)$, the running minimum $\tilde{\rho}_0(k)$ of the potential flows to infinity such that the dimensionful renormalized minimum — the magnetization — acquires a finite positive value (broken phase). For small initial $\tilde{\rho}_0(\Lambda)$, the running minimum $\tilde{\rho}_0(k)$, and thus the magnetization, vanishes at a finite scale t_s (symmetric phase). In between, there exists a critical initial parameter $\tilde{\rho}_0^{cr}(\Lambda)$ for which the running potential reaches a fixed point [35]. The critical exponent η is then given by the fixed point value η^* of the running anomalous dimension η_k [12, 35]. The critical exponent ν — describing the divergence of the correlation length near criticality — is related to the negative eigenvalue of the stability matrix, corresponding to the linearized flows of the renormalization functions on the field mesh, in the vicinity of the fixed point.

Setting $z_k(\tilde{\rho}) = 1$ for all k, *i.e.* neglecting the field renormalization constitutes the Local Potential Approximation (LPA). Within the LPA, the anomalous dimension η remains zero. Including a running field renormalization coefficient $Z_{k,0}$, independent of the field, allows one to refine the LPA by providing a non-trivial — although rough — determination of η. This approximation is referred to as First Order Approximation (FOA) in the following. The next step then consists in incorporating the full field-dependence of the renormalization function $z_k(\tilde{\rho})$, which is here called Second Order Approximation (and denoted SOA).

We successively study these three levels of approximations, and analyze for each the influence of R_k. We consider an exponential cutoff — which achieves an efficient separation of the low- and high-energy modes [14] — parametrized by a free amplitude α:

$$r(y) = \alpha\, \frac{1}{e^y - 1}, \tag{5}$$

where $r(y) = R_k(q^2)/(Z_{k,0}\, q^2)$ and $y = q^2/k^2$.

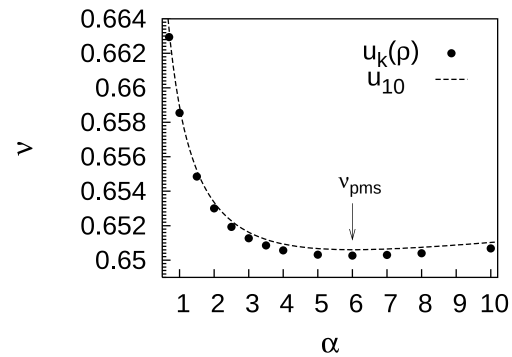

FIG. 1: Variations of ν with α at the LPA. The dashed line represents the results obtained with a tenth order truncated potential [29], dots those obtained keeping the full functional dependence of $u_k(\tilde{\rho})$ (present work).

For each parameter α, the initial value of $\tilde{\rho}_0(\Lambda)$ is fine-tuned to reach the critical regime and the associated exponents are computed. The optimal α is then determined through the PMS, that is for a specific exponent χ, the parameter α_{PMS} for which χ is stationary is selected:

$$\left.\frac{d\,\chi(\alpha)}{d\,\alpha}\right|_{\alpha_{\text{PMS}}} = 0, \tag{6}$$

and $\chi_{\text{PMS}} \equiv \chi(\alpha_{\text{PMS}})$ is identified as the optimized exponent. The hope is that imposing a constraint satisfied by χ in the exact theory improves the approximate determination of this quantity. The obvious drawback of this criterion is that Eq. (6) may exhibit many solutions. But in fact, this can be easily circumvented by complementing the PMS with a few basic rules that allow to simply discriminate between multiple solutions [29, 30].

We can now set out the results, which are collected in Table I along with the best theoretical determinations. Figure 1 displays the variations of ν with the parameter

α within the LPA, together with the analogous curve ensuing from the field truncation of the potential (at $p_u = 10$) [29]. Both $\nu(\alpha)$ curves — with and without field truncation — appear to coincide within a few tenths of percents. This agreement confirms the convergence of the field expansion and thus validates the former analysis [29]. Moreover, the function $\nu(\alpha)$ exhibits a unique extremum, which hence embodies the stationary solution of the PMS. The corresponding optimal exponent is $\nu_{\rm PMS} = 0.6503$.

Let us emphasize that within the LPA, $\nu(\alpha)$ always overvalues the "expected" exponent $\nu = 0.6304(13)$ (according to the best theoretical estimates), and that $\nu_{\rm PMS}$ corresponds to the minimum. Thus, as already highlighted in [29], the PMS selects the most accurate exponent achievable within a given approximation and is therefore equivalent to optimizing the precision.

		ν		η	
		full	truncated	full	truncated
	LPA	0.6503	0.6506	0	0
(a)	FOA	0.6291	0.6267	0.1058	0.1058
	SOA	0.6277	0.6281	0.0443	0.0443
(b)	7-loop	0.6304(13)		0.0335(25)	
(c)	MC	0.6297(5)		0.0362(8)	

TABLE I: Critical exponents of the three-dimensional Ising model: *a*) effective average action method: "full" stands for results keeping the full field-dependence of u_k and z_k (present work), "truncated" denotes results from field expansions of these functions [29]; *b*) 6-loop calculations including 7-loop corrections [36]; *c*) Monte-Carlo simulations [37].

We then come to the FOA, for which the anomalous dimension becomes nontrivial. The variations of ν and η with the parameter α are displayed in Figure 2, compared with the curves following from a truncated potential (at $p_u = 10$) with a field renormalization coefficient $Z_{k,0}$ (which corresponds to $p_z = 0$ in Eq. (4)) [29]. Both approaches yield very similar results. Let us note that, for both, the anomalous dimension is defined by the fixed point value of η_k evaluated at the minimum of the potential (see Eq. (3)). Hence the corresponding results are not expected to differ much. Indeed, Figure 2 shows that $\eta(\alpha)$ computed from both procedures coincide. On the other hand, whether or not a field truncation is implemented alters the determination of ν. Only values of the potential and its derivatives *at* the minimum enter the computation of ν when the latter is field expanded, whereas, when integrating the field-dependent flows, the determination of this exponent mixes values on the whole field mesh considered and thus incorporates a richer information. However, the $\nu(\alpha)$ curves from both procedures also reveal a good agreement, below the percent level (see Figure 2 and Table I). Hence the smallness of the discrepancy seems to indicate that the vicinity of the minimum already captures most of the relevant features.

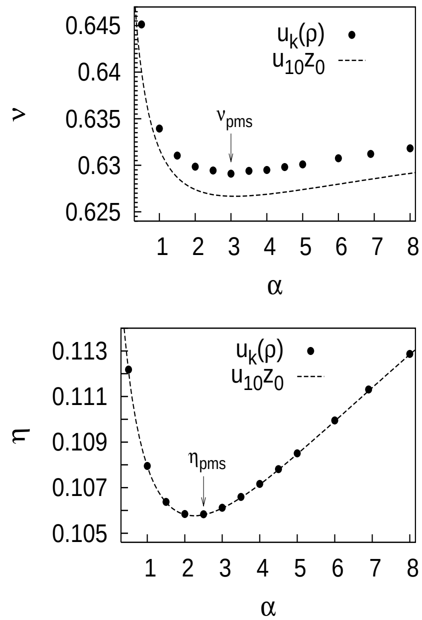

FIG. 2: Variations of ν and η with α at the FOA. The dashed line represents results obtained with a tenth order truncated potential and a field renormalization coefficient [29], dots those obtained keeping the full functional dependence of $u_k(\tilde\rho)$ (present work).

We can now discuss the optimization at the FOA. Both $\nu(\alpha)$ and $\eta(\alpha)$ still exhibit a single extremum and thus a unique PMS solution: $\nu_{\rm PMS} = 0.6291$ and $\eta_{\rm PMS} = 0.1058$. Again, these exponents turn out to minimize the distance to the best theoretical estimates (see Table I), since the latters lie below all $\nu(\alpha)$ and $\eta(\alpha)$ respectively and the PMS solutions are minima. The same conclusion can therefore be reiterated as for the equivalence between the PMS and the optimization of the accuracy.

Let us finally investigate the SOA, considering the full field-dependence of the field renormalization function $z_k(\tilde\rho)$. This order involves a considerable numerical task [39]. Figure 3 shows the variations of the two critical exponents with α, together with the analogous curves obtained by field expanding both $u_k(\tilde\rho)$ and $z_k(\tilde\rho)$ (to $p_u = 10$ and $p_z = 9$ respectively) [29]. First, these curves support the previous discussion as for the two procedures — with and without truncation — as the anomalous dimensions exactly match, and the ν exponents lie within a percent. Moreover, at this order, the $\nu(\alpha)$ curvature is reversed such that the unique extremum becomes a maximum. However, $\nu(\alpha)$ turns out to underestimate the "expected" exponent for all α, and thus the PMS solution still minimizes the distance to the best theoretical value. Indeed, $\nu_{\rm PMS} = 0.6277$ appears in close agreement with the 7-loop estimate $\nu = 0.6307(13)$. On the other hand, $\eta(\alpha)$ always

overvalues the "expected" anomalous dimension and $\eta_{\rm PMS} = 0.0443$ achieves the minimum. Thus, the PMS once again selects the most accurate exponents at a given truncation.

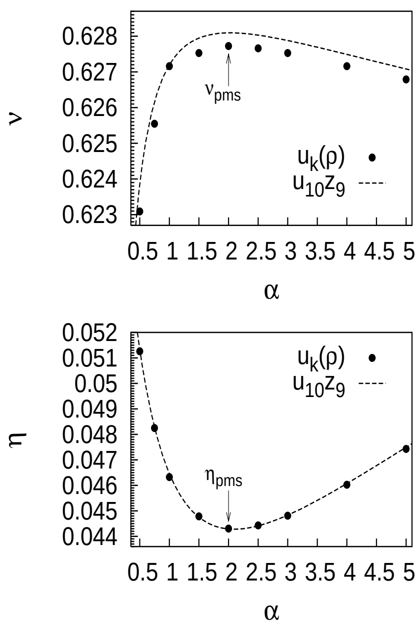

FIG. 3: Variations of ν and η with α at the SOA. The dashed line represents results obtained with a tenth order truncated potential and a ninth order truncated field renormalization function [29], dots those obtained keeping the full functional dependence of both $u_k(\tilde{\rho})$ and $z_k(\tilde{\rho})$ (present work).

We have proposed an optimization procedure applicable when dealing with field-dependent NPRG flows. The PMS has indeed appeared as an efficient tool, allowing to optimize the accuracy of the exponents of the three-dimensional Ising model, and thus constitutes a useful mean of control of the NPRG formalism even when field expansions are not possible or not desirable. This procedure would be particularly valuable to confirm the results obtained at the next order (∂^4) of the derivative expansion [30], since verifying explicitely the convergence of the field expansion at this order becomes extremely tedious. Furthermore, such a procedure would ensure one to control the influence of R_k and thus to work out reliable results within physical contexts requiring a functional description — for instance to deal with non-analytical potentials.

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[38] In Eq. (1), Tr stands for trace over internal indices and integration over internal momenta.

[39] The obtention of a single point of Figure 3 demands on average 35 days CPU of a 3.2GHz Pentium IV.